CYCLIC SETS IN MULTIDIMENSIONAL VOTING MODELS

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A series of papers has established the lack of stability and consistency of majority-rule outcomes within the context of multidimensional policy spaces. In a seminal paper, Plott (1967) showed that a median point exists only when severe restrictions are placed on the distribution and preferences of voters within the policy space. Kramer (1976) and Wagstaff (1976) demonstrated that majority-rule produces some intransitivities or cycles unless preferences are essentially equivalent. Schofield (1976) and Matthews (1977) derived local cycling results for a general class of utility functions. Furthermore, McKelvey (1977) and Cohen and Matthews (1977) have explored the possibility of constructing agenda that allow the voters to move from a specified point in the space to another point by a sequence of majority-rule decisions. ¹

These results indicate that some majority-rule cycles are inevitable, and that a median rarely exists. The top cycle set is thus an important solution concept, since it allows both the absence of a median and the presence of cycling. The set, characterized by Schwartz (1976), is a cycling subset of the policy space. Points in the set dominate all points outside the set. The top cycle set is a stable equilibrium since voters will not choose to move from a point or outcome inside the set to a point outside the set. However, the voters can move from any point in the set to any other point in the set by a sequence of majority rule decisions.

The top cycle set is only useful as a solution concept if it is "small". McKelvey (1976) proved that if voters have Euclidean preferences and distinct ideal points, and if no median exists, then the whole policy space is contained in the same cycling set. The result questions the appropriateness of a top cycle solution concept.

In this paper the properties of the top cycle set are examined for the class of convex voter preferences. Existence and uniqueness of the top cycle set are established in section 1. In the second section, the framework derived in section 1 is used to examine the size of the top cycle set. While McKelvey's result does not hold in general, his theorem is extended to preferences which conform to certain regularity conditions. Ellipsoidal preferences satisfy these conditions, and thus result in a top cycle set that encompasses the entire policy space.

* I wish to thank John Ferejohn and my colleagues in the 1977 Social Science workshop. I especially thank Charles Plott and Richard McKelvey for commenting on earlier versions.
Section 1

The theorems in this paper are developed within the paradigm of a multi-dimensional policy space represented by $\mathbb{R}^m$, and voters who vote in accordance with preferences defined on $\mathbb{R}^m$.

Formally, let:

- $\mathbb{R}^m$ = the policy space
- $N = \{1, \ldots, n\}$ = the set of voters

Possible outcomes are represented by points $x, y$ in $\mathbb{R}^m$. Each voter $i$ in $N$ has a continuous utility function $u^i : \mathbb{R}^m \rightarrow \mathbb{R}$. Voter $i$'s preference and indifference relations, $\succ_i$ and $\sim_i$ respectively, are defined over pairs of points in $\mathbb{R}^m$ by:

\[
\begin{align*}
& x \succ_i y \iff u^i(x) > u^i(y) \\
& x \sim_i y \iff u^i(x) = u^i(y)
\end{align*}
\]

The symbol $\succ_i$ denotes a "preferred or indifferent to" relation.

Let $W$ be the collection of subsets of $N$ which are winning coalitions, and $M$ be an element of $W$. Define the relation $\succ_M$ by:

\[
x \succ_M y \iff x \succ_i y \quad \forall i \in M
\]

If $x \succ_M y$ for some $M$ in $W$, then the voters will choose to move from $y$ to $x$. In this case, $x$ is said to dominate $y$. A simple majority decision rule is used, so $M$ is a winning coalition if more than half the voters are in $M$:

\[
M \in W \iff \left|\left|M\right|\right| \geq \frac{n + 1}{2}
\]

A point $x^*$ is dominating if it dominates every other point in $\mathbb{R}^m$:

\[
\forall x \in \mathbb{R}^m, \exists M \in W \ni x^* \succ_M x
\]

A point $x^*$ is undominated if no point in $\mathbb{R}^m$ dominates $x^*$.

A point $y$ can be reached from a point $x$ if it is possible to move from $x$ to $y$ by a sequence of majority rule decisions.

Formally, $y$ can be reached from $x$ if there exist points $x_1, \ldots, x_p$ in $\mathbb{R}^m$ such that

\[
(1) \quad x_1 = x, \quad x_p = y \\
(2) \quad \forall j, i = 2, \ldots, p \exists M_j \in W \ni x_j \succ_M x_{j-1}
\]

If $x$ can be reached from itself, then a cycle exists. A cycling set is a subset $S$ of $\mathbb{R}^m$ such that any point in $S$ can be reached from any other point in $S$ by a sequence of points all of which are contained in $S$. The top cycle set $V$ is the smallest cycling set where every point in $V$ dominates every point not in $V$. In order to formalize the concept of the top cycle set, the set of points that can be reached from $x$ (denoted $N_x^\infty$) is derived:

Let $x \in \mathbb{R}^m$. Define:

\[
N_x^1 = \{ y \in \mathbb{R}^m | y \succ_M x \text{ for some } M \in W \} \\
N_x^2 = \{ y \in \mathbb{R}^m | \exists z \in N_x^1, M \in W \ni y \succ_M z \} \\
\vdots \\
N_x^k = \{ y \in \mathbb{R}^m | \exists z \in N_x^{k-1}, M \in W \ni y \succ_M z \}
\]

\[
N_x^\infty = \bigcup_{k = 1}^\infty N_x^k
\]

By construction, any point in $N_x^\infty$ can be reached from $x$ by a sequence of points contained entirely in $N_x^\infty$. 
The top cycle set \( V \) is the subset of \( \mathbb{R}^m \) such that for all \( x \) in \( V \), \( N_x = V \). The remainder of this section is concerned with demonstrating that \( V \) exists, is unique, and has the desired characteristics. Lemmas 1 - 6 establish properties of \( N_x \) for an arbitrary \( x \) in \( \mathbb{R}^m \), and Theorem 1 addresses the top cycle set directly.

The proofs assume the following:

Assumption 1: The preference orderings are strictly quasi-convex:
\[
\forall i \in N, x \not\in \mathbf{1} y \implies \forall \lambda, 0 < \lambda < 1, \lambda x + (1 - \lambda) y \not\in \mathbf{1} y
\]

Assumption 2: The number of voters is odd and greater than 2.

Assumption 3: As the norm of \( x \) approaches infinity, the voters' utility functions approach either a lower bound \( L \), or \(-\infty\):
\[
\|x\| \rightarrow +\infty \implies \begin{cases} 
1 \quad U^i(x) \rightarrow L, \text{ where } L = \inf_{x \in \mathbb{R}^m} U^i(x) \\
0 \quad U^i(x) \rightarrow -\infty
\end{cases}
\]

Assumption 4: \( \mathbb{R}^m \) contains no dominating point.

Lemma 1: For all \( x \in \mathbb{R}^m \), \( N_x \) is non-empty.\(^3\)

Lemma 1 is equivalent to the statement that for all \( x \) in \( \mathbb{R}^m \) there exists a \( y \) in \( \mathbb{R}^m \) and \( M \in W \) such that \( y \succ_M x \). The convexity restriction on preferences and the lack of a dominating point imply that no point is undominated. This lemma indicates that if the top cycle set \( V \) is a set \( N_x \) for some \( x \) in \( \mathbb{R}^m \), then \( V \) is non-empty.

Lemma 2: \( N_x \) contains no points that are undominated in \( N_x \), and contains no point which does not dominate any other point in \( N_x \), i.e., \( \forall y \in N_x, \exists z, w \in N_x \) and \( N_1, N_2 \in W \) such that \( y \not\succ_M z, w \not\succ_M y \).

Lemma 3: \( N_x \) is either the entire space \( \mathbb{R}^m \), or is a bounded, convex, open subset of \( \mathbb{R}^m \).

If \( N_x \) is unbounded for all \( x \) in \( \mathbb{R}^m \), then Lemma 3 indicates that every point in \( \mathbb{R}^m \) can be reached from every other point in \( \mathbb{R}^m \). The entire space is contained in the same cycling set, and the top cycle set \( V \) is equal to \( \mathbb{R}^m \). This is equivalent to McKelvey's result (1976). The majority decision rule then places no constraints on possible outcomes. Thus, a further assumption is incorporated for the remainder of this section:

Assumption 5: \( N_x \) is bounded.

\( B \) denotes the boundary of \( N_x \).

Lemma 4: Points on the boundary \( B \) of \( N_x \) are not comparable, i.e., \( \forall y, z \in B \) and \( \forall M \in W \),
\[
\not\exists y \succ_M z \text{ and } \not\exists z \succ_M y
\]

Lemma 5: Every point in \( N_x \) dominates every point not in \( N_x \):
\[
\forall y \in N_x, \forall z \not\in N_x \exists M \in W \forall y \succ_M z
\]

Lemma 6: Every point in \( B \) dominates every point which is neither in \( N_x \) nor in \( B \):
\[
\forall y \in B, \forall z \in \mathbb{R}^m - N_x \implies (\mathbb{R}^m - (N_x \cup B)), \exists M \in W \forall y \succ_M z
\]

Lemmas 4 - 6 characterize the equilibrium aspects of the top cycle set. If for some \( x^* \), \( N_{x^*} \) is a cycling set, and it is the
smallest of the collection of sets $\{N_x \mid x \in \mathbb{R}^m \}$ then the
desired result is derived. Thus, the top
cycle set is a natural solution concept for the majority decision
rule.

Theorem 1: There exists an open set $V \subseteq \mathbb{R}^m$ such that:

1. $V \neq \emptyset$
2. $\forall x \in V, y \neq V \exists x \in \mathbb{R}^m y$
3. $\forall x, y \in V \exists x_1, \ldots, x_p$ such that
   i. $x_1 = x, x_p = y$
   ii. $x_j \geq x_{j-1}$ for some $x_j \in W, j = 2, \ldots, p$
4. $V$ is unique

Proof:
If there is no dominating point and if the entire policy space is
not contained in the same cycle set, then by Lemma 1 there must
exist $x$ in $\mathbb{R}^m$ such that $N_x$ is bounded and nonempty. We fix $N_x$ and
consider the set of points contained in the closure of $N_x$, denoted
by $\{y \mid y \in \overline{N}_x\}$. Each $y$ in $\overline{N}_x$ is associated with a set $N_y$.

Claim: For all $y \in \overline{N}_x, N_y \subseteq N_x$. If $y \in \overline{N}_x$, then either $y \in N_x$
or $y \in B$. If $y \in N_x$, then $y$ can be reached from $x$, say by the
sequence $x_1, \ldots, x_q$. Then if $z$ can be reached from $y$, by $x_{q1}, \ldots, x_{q'}$,
z can be reached from $x$ by the sequence $x_1, \ldots, x_p$. Thus $N_y \subseteq N_x$.

If $y \in B$, by Lemma 6 only points in $N_x$ dominate $y$. then
$$N_y = \{z \mid \exists x \in \mathbb{R}^m \geq x \forall y \subseteq N_x\}$$

and hence $N_y$ must again be contained in $N_x$. Thus, for all $y$ in $\overline{N}_x$,
$N_y$ is bounded, satisfies Lemmas 1-6, and is contained in $N_x$. If
$N_y \subseteq N_x$, then $\overline{N}_y \subseteq \overline{N}_x$.

Let
$$Z = \{\overline{N}_y \mid y \in \overline{N}_x\}$$

Claim: $Z$ is preordered by the inclusion relation $\supseteq$, i.e. for all $\overline{N}_y \in Z$
$\overline{N}_y \subseteq \overline{N}_x$, either $\overline{N}_y \supseteq \overline{N}_x$ or $\overline{N}_z \supseteq \overline{N}_y$.

Let $y^* \in \overline{N}_y$ and $z^* \in \overline{N}_z$ and suppose that $y^* \notin \overline{N}_z$ and $z^* \notin \overline{N}_y$. Then
by Lemmas 5 and 6, there exist winning coalitions $M_1$ and $M_2$ such that
$$y^* \supseteq M_1 \text{ and } z^* \supseteq M_2 \text{ } y^*.$$ 

Since $\|N\| > \frac{m+1}{2}$ for all $M$ in $W$, this is impossible. Thus either
$\overline{N}_y \supseteq \overline{N}_z$ or $\overline{N}_z \supseteq \overline{N}_y$.

A collection of sets has the finite intersection property if the
intersection of elements of every finite subcollection is nonempty.
Since $Z$ is preordered by inclusion, it must have the finite intersection
property. To see this, note that any finite subcollection of $Z$, say
$(\overline{N}_{y_1}, \ldots, \overline{N}_{y_n})$, must contain a "smallest" member. Formally, we can
choose $i \in \{1, \ldots, n\}$ such that
$$\overline{N}_{y_1} \subseteq \overline{N}_{y_j} \text{ for all } j \in \{1, \ldots, n\}.$$ 

Thus
$$\bigcap_{j=1}^n \overline{N}_{y_j} = \overline{N}_{y_1} \neq \emptyset.$$
The collection $Z$ is a collection of closed sets each of which is contained in $\bar{N}_x$, a closed and bounded set which is thus compact. A property of compact sets is that every collection of closed subsets with the finite intersection property must have nonempty intersection. In other words, if

$$\bigcap_{y \in \bar{N}_x} \bar{N}_y = \emptyset,$$

then there exists $y_1, \ldots, y_n \in \bar{N}_x$ such that

$$\bigcap_{i \in \{1, \ldots, n\}} \bar{N}_{y_i} = \emptyset.$$

Since $Z$ has the finite intersection property,

$$\bar{V} = \bigcap_{\bar{N}_y \in Z} \bar{N}_y \neq \emptyset.$$

Let $x^* \in \bar{V}$.

By construction of the sets $\bar{N}_y$, since $x^* \in \bar{N}_y$, $\bar{N}_{x^*} \subseteq \bar{N}_y$ for all $\bar{N}_y \in Z$, and hence $\bar{N}_{x^*} \subseteq \bar{V}$.

Furthermore $\bar{N}_{x^*}$ must be equal to $\bar{V}$, since $x^* \in \bar{N}_x$ and $\bar{N}_{x^*} \in Z$. Thus:

$$\bar{V} = \bigcap_{\bar{N}_y \in Z} \bar{N}_y = \left( \bigcap_{\bar{N}_y \in Z} \bar{N}_y \right) \cap \bar{N}_{x^*} \subseteq \bar{N}_{x^*}.$$

By identical reasoning, we note that for all $z$ contained in $\bar{V}$,

$$\bar{N}_z = \bar{V}.$$
Section 2

In the previous section it was shown that subject to the first four assumptions stated there, either the entire space
cycles or a well-defined bounded set exists which satisfies the
definition of the top cycle set. Points on the boundary of this
set are incomparable. When does the boundary not exist? That is
when is V the entire space?

McKelvey (1976) proves that $V = R^m$ if voters have Euclidean
preferences. He conjectures that the result holds for more general
utility functions. However, the boundary does exist in some cases
when voters have strictly quasi-convex preferences. One case is
shown in Figure 1, where all voters' indifference contours converge
to the same contour B. However, $V = R^m$ if voters have elliptical
preferences, that is, for each i, there is a positive definite
matrix $A_i$ and an ideal point $x_i^*$ such that:

$$x \succ_i y \iff (x - x_i^*)' A_i (x - x_i^*) < (y - x_i^*)' A_i (y - x_i^*)$$

Theorem 2 establishes the non-boundedness of V when preferences
satisfy Assumptions 1 - 4 in section 1 and the following:

Assumption 6: Suppose $x \succ_i y$. Then for all $\beta > 0$,

$$x + \beta (x - y) \succ_i y + \beta (y - x)$$

The assumption is illustrated in Figure 2. If $y = -\alpha x$ for some
$\alpha > 0$, then Assumption 6 states that for all positive $\beta$,

$$x \succ_i -\alpha x \rightarrow \beta x \succ_i -\beta \alpha x$$

Theorem 2: Let $N = \{1, \ldots, n\}$ be the set of voters, and $R^m$ be the
policy space. Suppose assumptions 1 - 4 and 6 hold. Then $V = R^m$.

Proof: By section 1, V must either be $R^m$ or some bounded, open,
convex, non-empty subset of $R^m$. Assume the latter holds, and let
B be the boundary of V. Let $z^*$ denote a point in V. Without loss
of generality, we can assume that $z^*$ is the origin, 0. By lemma
1, there is some point y in $R^m$ and a winning coalition $M^*$ such that:

$$y \succ_{M^*} 0$$

Since by Theorem 1, $N_0 = V$, y must be an element of V. Let $i \in M^*$.

Then $y \succ i 0$

Claim: $0 \succ i -\alpha y \forall \alpha > 0$

Proof: Suppose $y \succ i -\alpha y$. Then $\forall \alpha > 0, \lambda y + (1 - \lambda) (-\alpha y) \succ i -\alpha y$

Let $\lambda = \frac{\alpha}{1 + \alpha}$

Then $(\frac{\alpha}{1 + \alpha}) y + (1 - \frac{\alpha}{1 + \alpha}) (-\alpha y) = 0 \succ i -\alpha y$

Similarly, if $-\alpha y \succ i y$, 0 $\succ i y$, which leads a contradiction.

Thus, $0 \succ i -\alpha y$ for all $\alpha > 0$ and for all i in M*.

Hence, $0 \succ M^* -\alpha y$.

Now as $0 \in V$, and V is open, $\exists \alpha > 0$ such that $-\alpha y \in V$. Furthermore, $y \in V$.

By assumption, V is bounded. There is then some positive real number $\beta$
so that $\beta y \in B$. Furthermore, we can choose $\alpha$ so that $-\alpha y \in B$.

Then by assumption 6, $\beta y \succ_{M^*} -\alpha y$

But Theorem 1 states that points on the boundary of V are not comparable.

The contradiction implies that B cannot exist, so $V = R^m$.

Corollary 1: Suppose voters have elliptical preferences, i.e.
there exist positive definite matrices $A_1, \ldots, A_n$ and points
$x_1^*, \ldots, x_n^*$ in $R^m$ such that

$$x \succ i y \iff (x - x_i^*)' A_i (x - x_i^*) < (y - x_i^*)' A_i (y - x_i^*)$$

and suppose n is odd and greater than 2 and no dominating point
exists. Then $V = R^m$. 
Thus the regularity assumptions on preferences yields the result that the top cycle space is the entire space. Elliptical preferences satisfy the assumptions of the theorem. Establishing the assumptions involve straightforward algebraic proofs, which are contained in the appendix.

Conclusions

In this paper an equilibrium solution concept - the top cycle set - was investigated. If majority rule places any constraints on possible solutions, then it constrains solutions to a well-defined set which exhibits the characteristics of the top cycle set. However Theorem 2 indicates that frequently the set may be the entire space. Thus it is reasonable to hypothesize that the outcomes in institutions characterized by majority rule are determined by factors exogenous to individual preferences or to the voting rule.

Appendix

Lemma 1: $N_x \neq \emptyset$

To show that $N_x$ is non-empty, it is sufficient to show that $N_x^1 = \{y \mid y \succeq_M x\}$ is non-empty. Since $x$ is not a dominating point, $y^* \in \mathbb{R}^m$ such that $x \succeq_M y^*$. Let:

$I = \{i \mid x \succ_i y^*\}$

$J = \{i \mid y^* \succ_i x\}$

$K = \{i \mid y^* \succ_i x\}$

$\exists x \succ_M y^* \rightarrow J \cup K \in \mathbb{W}$. For all $i \in J$, there exists an open set $O^i(y^*)$ such that for all $y \in O^i(y^*)$, $y \succ_i x$. For all $i \in K$, and $\lambda, 0 < \lambda < 1, \lambda y^* + (1 - \lambda) x \succ_i x$

Choose $\lambda$ sufficiently close to 1 that $y^{**} = \lambda y^* + (1 - \lambda) x \in O^i(y^*)$ $\forall i \in J$.

Then $y^{**} \succ_i x$ for all $i \in J \cup K$, and thus $y^{**} \succeq_M x$, and $y^{**} \in N_x^1$

Lemma 2: $N_x$ has no undominated or pessimal points in $N_x$.

Lemma 1 implies that $N_x$ has no undominated points in $N_x'$, since for all $y \in \mathbb{R}^m, N_y \neq \emptyset$, and if $y \in N_x$, by definition of $N_x'$, $N_y \subseteq N_x$.

To show the second part, note that by the construction of $N_x'$, for all $y \in N_x'$, either $y$ dominates some other point in $N_x$, in which case we are done, or $y \succeq_M x$. Suppose $y \succeq_M x$. Then there exists a $\lambda, 0 < \lambda < 1$, such that $y \succeq_M \lambda y + (1 - \lambda) x \succeq_M x$.

Lemma 3: $N_x$ is a bounded open convex set or the entire space.

Suppose first that $N_x$ is unbounded and suppose that $y$ is not contained in $N_x$. By Lemma 2, there exists $M \in \mathbb{W}$, and $y^* \in \mathbb{R}^m$ such that $y \succeq_M y^*$. Thus for all $i \in M$,

(a) $U^i(y) > U^i(y^*)$
Assumption 3 states that as \( \| z \| \to +\infty \), \( U^i(z) \) approaches either

\(-\infty\), or a lower bound. In either case, for all \( i \in M \),

\( U^i(y) > U^i(z) \) by (a). If \( N^i_x \) is unbounded, we can choose an element

\( z \in N^i_x \) such that the norm of \( z \) is arbitrarily large, so

\( U^i(y) > U^i(z) \)

for all \( i \in M \), \( \Rightarrow y \in N^i_x \), and thus, \( N^i_x = \mathbb{R}^m \).

Now suppose that \( N^i_x \) is bounded. Since preferences are continuous, \( N^i_x \) must be open, so it remains only to be shown that \( N^i_x \) is convex.

Let \( y, z \in N^i_x \). Either \( y \) does not dominate \( z \) or vice-versa. Suppose

that \( y \) does not dominate \( z \):

\[ \forall M \in W, \quad \nexists \ y, y \succ_M z. \]

Let:

\[ J = \{ i \mid z \succ_i y \} \]

\[ K = \{ i \mid y \succ_i z \} \]

Then \( J \cup K \subseteq M \).

By assumption 1, for all \( \lambda, 0 < \lambda < 1 \), and for \( i \in J \cup K \),

\[ \lambda z + (1 - \lambda) y \succ_i y. \]

Thus, for all \( \lambda, 0 < \lambda < 1 \), \( \lambda z + (1 - \lambda) y \in N^i_x \).

Lemma 4: For all \( y, z \in B, M \in W, \quad \nexists \ y, y \succ_M z \) and \( \nexists \ z, z \succ_M y \).

Suppose there exist \( y, z \in B \) and \( M \in W \) such that \( y \succ_M z \). Then for

all \( i \in M, \quad z \succ_i y \). For each \( i \), \( U^i \) is continuous, and \( U^i(y) > U^i(z) \).

Thus there exists an open neighborhood \( O^i(z) \) such that for all \( w \in O^i(z) \),

\( y \succ w \). Let \( O(z) = \cap_{i \in M} O^i(z) \). As \( z \in B, O(z) \cap N^i_x \neq \emptyset \).

Let \( w \in O(z) \cap N^i_x \). Then \( y \succ w \) for all \( i \in M, \Rightarrow y \in N^i_x \).

But since \( N^i_x \) is open, \( y \) cannot be in \( N^i_x \). Thus, \( \nexists \ y \succ_M z \).

Lemma 5: For all \( y \in N^i_x, z \notin N^i_x \), there exists \( M \in W \) such that

\( y \succ_M z \).

Suppose that for all \( M \in W, \quad \nexists \ y \succ_M z \). Then by the same argument as

in Lemma 3, there are points \( w \) arbitrarily close to \( y \) and \( M \in W \) such that \( z \succ_M w \). Since \( N^i_x \) is open, we can choose \( w \in N^i_x \). Thus,

\( z \notin N^i_x \).

Lemma 6: For all \( y \in B, z \notin N^i_x \), there exists an \( M \) such that \( y \succ_M z \).

Suppose not. If there exists \( M \in W \) such that \( z \succ_M y \), then there is an open neighborhood \( O(y) \) such that for all \( w \in O(y), z \succ_M w \).

Since \( O(y) \cap N^i_x \neq \emptyset, z \succ_M w \) for all \( w \in N^i_x \).

If for all \( M \in W, \quad \nexists \ y, \succ_M y \) and \( \nexists \ z, \succ_M z \), then there are points \( w \)

arbitrarily close to \( z \) such that \( w \succ_M y \) (see the argument in Lemma

4) and \( \mathbb{R}^m - N^i_x \) is open, \( w \in \mathbb{R}^m - N^i_x \). But then \( w \notin N^i_x \) by the

argument above in Lemma 6.

Thus for all \( y \in B, z \notin N^i_x \), there exists \( M \in W \) such that \( y \succ_M z \).

Corollary: Elliptical preferences satisfy Assumptions 1–4 and 6.

We prove here only assumption 6: the remainder is left as an exercise for

the interested reader. Let \( A \) be positive definite and \( \overline{x} \in \mathbb{R}^m \). Suppose

\( (x - \overline{x})' A (x - \overline{x}) < (y - \overline{x})' A (y - \overline{x}) \)

Let \( \beta > 0 \).

\( (x + \beta(y - x) - \overline{x})' A (x + \beta(y - x) - \overline{x}) \)

\( = (x - \overline{x})' A (x - \overline{x}) + \beta^2 (x - y)' A (x - y) \)

\( + \beta(x - \overline{x})' A (x - x + \overline{x} - y) + \beta(x - x + \overline{x} - y)' A (x - \overline{x}) \)

\( = (x - \overline{x})' A (x - \overline{x}) + \beta^2 (x - y)' A (x - y) + \beta(x - \overline{x})' A (x - \overline{x}) \)

\( - \beta(x - \overline{x})' A (y - \overline{x}) + \beta(x - \overline{x})' A (y - \overline{x})' A (x - \overline{x}) \)

\( = (x - \overline{x})' A (x - \overline{x}) + \beta^2 (x - y)' A (x - y) + \beta(x - \overline{x})' A (x - \overline{x}) \)

\( - \beta(x - \overline{x})' A (y - \overline{x}) + \beta(x - \overline{x})' A (y - \overline{x})' A (x - \overline{x}) \)
\[< (y - \bar{x})' A (y - x) + \beta^2 (y - y) ' A (y - x) + \beta (y - \bar{x})' A (y - \bar{x}) - \beta (y - \bar{x})' A (x - \bar{x}) + \beta (y - x) ' A (y - x) - \beta (x - \bar{x})' A (y - \bar{x}) = (y + \beta (y - x) - \bar{x})' A (y + \beta (y - x) - \bar{x})\]

Footnotes

1. McKelvey's paper came to my attention after the completion of the original version of this paper. His method of proof and some of his results are similar to those of this paper. The paper by myself and Steven Matthews was written after this paper was submitted for publication. It contains an extension of the results presented here.

2. Assumption 3 is only used to establish that the set of points in \(R^m\) which are preferred to \(x\) (called \(N_x\) in the text) is either bounded or is the entire space. As this proposition is central to Theorem 1, Assumption 3 is assumed in general.

3. The proofs of the lemmas are contained in the Appendix.

4. See Dugundji p. 32 for a statement of Zorn's Lemma and definitions.
References:


