ABSTRACT

The introduction of an additional player to serve as coordinator in an N-person abstract economy leads in a natural way to an N+1-person noncooperative game. Sufficient conditions on the abstract economy are considered which lead to the existence of equilibrium in the resulting game and hence for the abstract economy.
0. Introduction.

Debreu [1952] introduced the concept of an abstract economy as a generalization of a noncooperative game (Nash [1951]). In a noncooperative game each player has a set of strategies available to him regardless of what strategies the other players select. The choice of strategies by all of the players determine an outcome and the players are all assumed to have preferences over outcomes. These preferences over outcomes then generate preferences over strategy vectors. An equilibrium is a strategy vector such that no player can change his strategy so as to yield a more preferred outcome. In an abstract economy the set of strategies available to a player may be a proper subset of strategies which depends on the choices of the other players. An equilibrium for an abstract economy is a strategy vector which is feasible for all players given everyone's choices and no player can alter his strategy so as to effect a more preferred outcome. Arrow and Debreu [1954] used Debreu's result on the existence of equilibrium for an abstract economy to find sufficient conditions for the existence of Walrasian equilibrium for a market economy. The reduction of a market economy to an abstract economy is accomplished by introducing an "auctioneer" or "market participant", a new player whose strategy set is the set of prices. The auctioneer's choice of price restricts each consumer's choice of consumption through the budget correspondence. Profits of producers also depend on the auctioneer's choice of price, but their set of available productions does not depend on the price.

The preferences of consumers were assumed to be representable by real valued utility functions by Arrow and Debreu. Sonnenschein [1971] showed that transitivity of preferences could be dispensed with for proving the existence of Walrasian equilibrium. Mas-Colell [1974] was further able to dispense with completeness of preferences. Gale and Mas-Colell [1975] present a proof of existence of equilibrium for noncooperative games without complete or transitive preferences and use it to establish existence of Walrasian equilibrium. Shafer and Sonnenschein [1975] present a proof of existence of equilibrium for abstract economies without complete or transitive preferences and Shafer [1976] uses this to prove the existence of Walrasian equilibrium under fairly general conditions.

The theorem of Gale and Mas-Colell on noncooperative games follows from Shafer and Sonnenschein's theorem on abstract economies, but not conversely. This suggests that the Shafer-Sonnenschein theorem is stronger than needed to prove the existence of Walrasian equilibrium. By imposing additional hypotheses on the feasible strategy correspondences, while somewhat weakening assumptions on preferences, we can convert an N-person abstract economy to an N+1-player noncooperative game. These extra assumptions were essentially introduced by Borglin and Keiding [1976], and are automatically satisfied by Walrasian budget correspondences. This is why Gale and
Mas-Colell were able to prove the existence of Walrasian equilibrium with their theorem. Borgen and Heeding reduced an abstract economy to a 1-person game. Our technique of expanding the player set has a neat interpretation in terms of stationary points of a tatonnement procedure.

1. Definitions.

An N-person noncooperative game is a tuple $G = (X_i, P_i)_{i=1}^N$, where for each $i$,

$$P_i : X \to X_i$$

is a possibly empty-valued correspondence,

where $X = \prod_{j=1}^N X_j$. An equilibrium for $G$ is a strategy vector $\bar{x} \in X$ such that for each $i$

$$P_i(\bar{x}) = \emptyset \quad \text{or} \quad \bar{x}_i \in P_i(\bar{x}).$$

Here $X_i$ is the set of strategies of player $i$ and $P_i(x)$ is interpreted in one of two ways. The set $P_i(x)$ may be the set of strategies which player $i$ strictly prefers to $x_i$ given $x_{i-1}, x_{i+1}, \ldots, x_N$. Then $P_i(x) = \emptyset$ means that there are no preferred strategies to $x_i$ given the others’ choices. Under this interpretation it never happens that $\bar{x}_i \not\in P_i(\bar{x})$. The second way we may interpret $P_i(x)$ is as the set of "best replies" to $x_{i-1}, x_{i+1}, \ldots, x_N$. Then $P_i(x)$ is never empty and $\bar{x}_i \in P_i(\bar{x})$ is the appropriate equilibrium notion. Our definition of equilibrium allows for $P_i$ to have one interpretation for some players and the other interpretation for the rest.

An N-person abstract economy is a tuple $E = (X_i, S_i, p_i)_{i=1}^N$, where for each $i$,

$$P_i : \prod_{j=1}^N X_j \to X_i$$

is a possibly empty-valued correspondence and

$$S_i : \prod_{j=1}^N X_j \to X_i$$

is a nonempty-valued correspondence.

Put $X = \prod_{j=1}^N X_j$. We will write $S_i : X \to X_i$, with the understanding that $S_i(x)$ does not depend on $x_i$.

An equilibrium for $E$ is a vector $\bar{x} \in X$ such that for each $i$,

$$\bar{x}_i \in S_i(\bar{x})$$

and

$$P_i(\bar{x}) \cap S_i(\bar{x}) = \emptyset.$$

Again $X_i$ is player $i$’s strategy set and $S_i(x)$ is the set of feasible strategies for $i$ given the choices $x_1, x_{i-1}, x_{i+1}, \ldots, x_N$. The set $P_i(x)$ is the set of strategies preferred to $x_i$ given everyone else’s choice. The equilibrium condition is that the vector be feasible for everyone and no one has a strategy which is both preferred and feasible.
2. Theorems.

The following theorem is a slight variation of a theorem of Gale and Mas-Colell [1975]. Denote by co $P_i$, the correspondence whose value at $x$ is the convex hull of $P_i(x)$.

**Theorem 1.** Let $G = (X_i, P_i)_{i=1}^n$ be a noncooperative game satisfying for each $i$,

(i) $X_i$ is a nonempty compact convex subset of a finite dimensional euclidean space.

(ii) Either

(a) co $P_i$ has an open graph in $X \times X_i$, and for each $x \in X$, $x_i \notin co P_i(x)$.

or

(b) $P_i$ is a continuous singleton-valued correspondence, i.e., a continuous function.

Then $G$ has an equilibrium.

**Proof.** The proof is virtually identical to that of Gale and Mas-Colell [1975, p.10] and so we only sketch the proof. If $P_i$ satisfies (ii.a) let $U_i = \{x : co P_i(x) \neq \emptyset\}$ and let $f_i$ be a continuous selection from co $P_i |_{U_i}$. Define $\gamma_i : X \to X_i$ via

$$\gamma_i(x) = \begin{cases} f_i(x) & x \in U_i \\ X_i & x \notin U_i \end{cases}$$

If $P_i$ satisfies (ii.b) set $\gamma_i = P_i$.

Then $\gamma = \prod \gamma_i$ satisfies the hypotheses of Kakutani's theorem and has a fixed point $\bar{x}$. By construction, if $P_i$ satisfies (ii.a) then co $P_i(x) = \emptyset$ so $P_i(x) = \emptyset$ and if $P_i$ satisfies (ii.b) then $x_i \in P_i(\bar{x})$.

Q.E.D.

Theorem 2 below is a modification of a theorem of Shafer and Sonnenschein [1975]. The assumptions on preferences have been slightly weakened and those on the feasibility correspondences strengthened, but in a way so as to remain useful for proving the nonemptiness of Walrasian equilibrium. The proof proceeds by converting the abstract economy to an $N+1$ person noncooperative game.

**Theorem 2.** Let $E = (X_i, S_i, P_i)_{i=1}^N$ be an abstract economy satisfying, for each $i$,

(i) $X_i$ is a nonempty compact convex subset of a finite dimensional Euclidean space.

(ii) $S_i$ has a closed graph in $X \times X_i$, the correspondence $x \mapsto int S_i(x)$ has an open graph in $X \times X_i$, and for each $x \in X$, $S_i(x)$ is compact, convex, and has nonempty interior (relative to $X_i$).

(iii) co $P_i$ has an open graph in $X \times X_i$. 

(iv) For each \( x \in X, x_i \notin \text{co } P_i(x) \).

Then \( E \) has an equilibrium.

**Remark.** Shafer and Sonnenschein assume that \( S_i \) is lower semi-
continuous as well as closed. We have strengthened this in (ii.a) to
assuming \( \text{int } S_i \) is a nonempty-valued correspondence with open graph.

Note that since \( S_i(x) \) is compact and convex with nonempty interior
that \( S_i(x) = \text{cl } (\text{int } S_i(x)) \), i.e., \( S_i(x) \) is topologically regular.

Note that interior here is relative to \( X_i \), not the underlying
Euclidean space.

Shafer and Sonnenschein assume that \( P_i \) has an open graph. We
have weakened this to assuming \( \text{co } P_i \) has an open graph. Assuming that
\( P_i \) has open sections (see Bergstrom, Parks and Rader [1976]), then
\( \text{co } P_i \) will have open graph. (In this case \( \text{co } P_i \) will be convex-valued
with open sections. See Bergstrom, Parks and Rader [1976] and Shafer
[1974].) The assumption of open sections for \( P_i \) is strictly weaker
than the assumption of open graph as long as \( P_i \) is not convex-valued.

**Proof of Theorem 2.** We define an \( N+1 \) person game as follows.

Put \( Z_0 = \prod_{i=1}^N X_i \), and for \( i = 1, \ldots, N \) put \( Z_i = X_i \). Set \( Z = \prod_{i=1}^N Z_i \). A
typical element of \( Z \) will be denoted \((x, y)\), where \( x \in Z_0 \) and
\( y \in \prod_{i=1}^N Z_i \). Define preference correspondences \( \mu_i : Z \rightarrow Z_i \) as
follows.

Define \( \mu_0 \) by

\[
\mu_0(x, y) = \{y\},
\]

and for \( i = 1, \ldots, N \) set

\[
\mu_i(x, y) = \begin{cases} \text{int } S_i(x) & \text{if } y \notin S_i(x) \\ \text{co } P_i(y) \cap \text{int } S_i(x) & \text{if } y \in S_i(x) \end{cases}
\]

Note that \( \mu_0 \) is continuous and singleton-valued and that for \( i = 1, \ldots, N \)
the correspondence \( \mu_i \) is convex-valued and satisfies \( y \notin \mu_i(x, y) \).

Also for \( i = 1, \ldots, N \), the graph of \( \mu_i \) is open. To see this for
\( i = 1, \ldots, N \) set

\[
A_i = \{(x, y, z) : z \in \text{int } S_i(x)\},
\]

\[
B_i = \{(x, y, z) : y \notin S_i(x)\},
\]

and

\[
C_i = \{(x, y, z) : z \notin \text{co } P_i(y)\},
\]

and note that

\[
\text{Gr } \mu_i = (A_i \cap B_i) \cup (A_i \cap C_i).
\]

The set \( A_i \) is open because \( \text{int } S_i \) has open graph and \( C_i \) is open by
hypothesis (iii). The set \( B_i \) is also open, for if \( y \notin S_i(x) \) then
there is a closed neighborhood \( F \) of \( y \) such that \( S_i(x) \subset F^c \) and upper
hemicontinuity of \( S_i \) then gives the desired result.

Thus the hypotheses of Theorem 1 are satisfied and so there exists \((x,y) \in Z\) such that

(a) \( \bar{x} \in P_0(x,y) \)

and for \( i = 1, \ldots, N \)

(b) \( \mu_i(x,y) = \varnothing \).

Now (a) implies \( \bar{x} = \bar{y} \) and since \( S_1(x) \) is never empty (b) becomes

\[
\text{co} \ P_1(x) \cap \text{int} \ S_1(x) = \varnothing \quad \text{for} \quad i = 1, \ldots, N.
\]

Thus \( P_1(x) \cap \text{int} \ S_1(x) = \varnothing \), but \( S_1(x) = \text{co} \ \text{int} \ S_1(x) \) and \( P_1(x) \) is open so \( P_1(x) \cap S_1(x) = \varnothing \), i.e., \( \bar{x} \) is an equilibrium.

Remark. We can view the N+1 game as a formalization of a tattonnement process. The N+1st player, call him the coordinator, performs in the abstract what a Walrasian auctioneer does in a market economy. The coordinator suggests a strategy vector. The players look at the choices they have made to see whether they are feasible given the coordinator's suggestion. If not, they choose a new strategy which is feasible. If the choice is feasible they try to improve upon it. Otherwise they don't change their strategy. The coordinator changes his suggestion to what the players were doing originally. A stationary point of this process is an equilibrium for the abstract economy. The question of under what circumstances this process is convergent remains to be investigated.

Note. We could have defined \( \mu_0 \) in an alternative fashion, namely

\[
\mu_0(x,y) = \{ z \in Z_0 : |z - y| < |x - y| \}
\]

Then \( \mu_0 \) would have an open graph and be convex valued with \( x \not\in \mu(x,y) \) so Theorem 1 would still apply.

Remark. In the application of Theorem 2 to the problem of existence of Walrasian equilibrium we have the following interpretations. Call the auctioneer player 1. Except for the auctioneer, each \( X_i \) is a subset of the commodity space \( \mathbb{R}^k \). Compactness is achieved by suitably truncating consumption and production sets. For the auctioneer \( X_1 \) is a subset of price space, also \( \mathbb{R}^k \). For producers \( S_i \) is a constant correspondence equal to their truncated convex production set. This satisfies (ii) as all interiors are relative to \( X_i \). Likewise for the auctioneer \( S_i \equiv X_i \). For consumers,

\[
S_i(p,x_2, \ldots, x_N) = \{ y_i \in X_i : p \cdot y_i \leq p \cdot w_i + \sum_j \theta_{ij} (p \cdot x_j)^+ \}
\]

where \( w_i \) is his endowment and \( \theta_{ij} \) is his share of firm \( j \). If player \( j \) is not a firm then \( \theta_{ij} = 0 \). Under the typical assumption (Shafer [1976], Gale and Mas-Colell [1975], Debreu [1959]) that \( w_i \in \text{int} \ X_i \) (or something like this), then \( S_i \) will satisfy the hypotheses of Theorem 2.
REFERENCES


