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SPECULATIVE HOLDINGS UNDER LINEAR EXPECTATION PROCESSES--
A MEAN-VARIANCE APPROACH

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SOCIAL SCIENCE WORKING PAPER 533

July 1984
Abstract

In this paper, we considered a discrete time abstract market model where the associated commodity is storable. Also, instead of assuming expected profit maximizing speculators, we assumed they employed mean-variance approaches.

Within this framework, given a non-degenerate quadratic inventory cost function and a linear expectation process, the optimal speculative carryover may be decomposed into four components of which two are special features arising from mean-variance considerations.

Furthermore, assuming a linear non-speculative excess demand function, Friedman's conjecture (i.e., profitable speculation necessarily stabilizes prices) holds from an ex ante point of view.
Speculative Holdings Under Linear Expectation Processes—
A Mean-Variance Approach

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I. Introduction

There have been several studies that attempt to characterize speculators' behavior under linear expectation processes [2] [4]. However, these papers assumed that speculators are expected profit maximizers, regardless of the riskiness of their market operations. In this paper, we assume that speculators employ a mean-variance approach, and then characterize their impacts on the market again assuming a linear expectation process. Within this framework, and assuming a linear non-speculative excess demand function, Friedman’s conjecture holds (i.e., profitable speculation necessarily stabilizes prices) from an ex ante point of view.

The plan of this paper is as follows: In section II, we describe the market structure and the speculator's problem; in section III and IV, dynamic programming is applied to solve the speculator's problem and some properties of the solution are exploited. In section V, linear expectation rules are introduced. We consider the special case when inventory cost is a fixed constant in section VI. The general case is dealt with in section VII.

Non-speculative excess demand is introduced in section VIII. Using this, we derive market price behavior and examine Friedman’s conjecture in section IX. Finally, section X states the conclusions.

II. Market Structure

Consider a discrete time spot market, where the associated commodity is storable. There is no forward or futures market in this commodity, and short-selling in the spot market is prohibited. The market opens at time \( t = 0, 1, 2, \ldots \) and transactions take place immediately thereafter.

There are three different types of agents in this market: producers, speculators and consumers. Producers and consumers as a group are called non-speculators. The type of each agent is exogenously determined. We also assume that the decisions of producers and consumers are made without considering the effects of speculators. Hence, we can treat non-speculative excess demand as exogenously given. Random effects that enter the model either come from the production side or the non-speculative demand side, but are assumed to be independent of speculators' behavior.

Each speculator takes prices as given (i.e., the case of competitive speculation) and he employs a mean-variance approach to solve his decision problem, using all information available to him. Let \( S_t \) denote the stock level at time \( t \) for a specified speculator (later, we'll assume all speculators are identical). Now, at time \( t \), the speculator observes the market price \( P_t \) and his carryover from the previous period \( S_{t-1} \). He then constructs a probability density function to summarize his expectations about next period's market
price $P_{t+1}$ using all available information. From this p.d.f., he determines his stock level $S_t$. Any inventory holding cost $h(S_t)$ is assumed to be incurred at time $t$.

Let $\beta$ be the discount factor employed by this speculator and let $\lambda/2$ be the weighting factor of market risk (variance) in his objective function. Then, the speculator entering the market at time $t$ solves the following problem:

\[
\text{(A)} \quad \max \sum_{i=t}^{\infty} \beta^{-i} \left[ \mathbb{E}[P_{i+1}(S_i - S_{i+1}) - h(S_{i+1})] - \frac{\lambda}{2} \text{Var}[P_{i+1}(S_i - S_{i+1}) - h(S_{i+1})] + P_t(S_{t-1} - S_t) - h(S_t) \right]
\]

where the expectations are taken conditional on his available information, hence they are different operators at different points in time. Speculators are assumed to be risk averse, hence $\lambda > 0$.

Before trying to solve (A), we make two further assumptions:

1. $\text{Var}[P_t P_{t-1}] = \sigma^2, \forall P_t, P_{t-1}, t = 0, 1, 2, \ldots$;

2. $h(S) = \frac{c(S - b)^2}{2} + d, \forall S \geq 0$. Assumption (2) incorporates a convenience yield effect, i.e., at stock level $b$, we achieve minimum inventory cost. If there is no convenience yield effect, then minimum inventory cost should be achieved at $S = 0$ which implies $b = 0$.\(^3\) (See [4]).

III. Optimal Speculative Carryover

To solve problem (A), assume that $\lim_{t \to \infty} S_t = b$ (equivalently, this says in the limit, the speculator will choose a minimum cost inventory stock level), and consider a decision beginning at $t = 0$.

Under some regularity assumptions, we can utilize dynamic programming to solve the speculator's problem. Specifically, assume at time $T$, the speculator's problem is over and his stock decision is $S_T^* = b, \forall t \geq T$. Therefore, at time $(T-1)$, his problem is:

\[
\text{(A1)} \quad \max \frac{\beta}{E[P_T(S_{T-1} - b)]} \frac{\lambda}{2} \text{Var}[P_T(S_{T-1} - b)] \nonumber \\
+ P_{T-1}(S_{T-2} - S_{T-1}) - \left( \frac{c(S_{T-1} - b)^2}{2} + d \right)
\]

(Note that, at $(T-1)$, $P_{T-1}$ and $S_{T-2}$ are both known.)

The first order condition for (A1) is

\[
\beta E_T = \beta \lambda \sigma^2 (S_{T-1} - b) - P_{T-1} - c(S_{T-1} - b) = 0
\]

\[
\Rightarrow S_{T-1}^* = \frac{\beta E_T - P_{T-1} - bc}{\beta \lambda \sigma^2 + c} + \frac{\frac{c(S_{T-1} - b)^2}{2} + d}{\beta \lambda \sigma^2 + c}
\]

where $E_T = \text{EP}_T$, the conditional expectation of $P_T$ using information about $P_{T-1}$, and $S_{T-1}^*$ is the optimal choice of $S_T$, $t = 0, 1, \ldots$

Next, at time $t = T - 2$, his problem becomes

\[
\text{(A2)} \quad \max \frac{\beta}{E[P_{T-1}(S_{T-2} - S_{T-1})]} \frac{\lambda}{2} \text{Var}[P_{T-1}(S_{T-2} - S_{T-1})] \nonumber \\
+ P_{T-2}(S_{T-3} - S_{T-2}) - \left( \frac{c(S_{T-2} - b)^2}{2} + d \right) + K_0
\]

where $K_0$ is a constant term independent of $S_{T-2}$.

Since
\[ \text{Var} (P_{T-1}(S_{T-2} - S_{T-1}^*)) = \frac{\text{Var} (P_{T-1}(S_{T-2} - \frac{\beta E_{T} - P_{T-1} + bc + b\beta \lambda^2}{\beta \lambda^2 + c}) =} \frac{\beta \lambda^2 + c}{2} \]

\[ \frac{\beta E_{T} - P_{T-1} - E(P_{T-1}(\beta E_{T} - P_{T-1})))}{\beta \lambda^2 + c} = \frac{\beta E_{T} - P_{T-1} - E(P_{T-1}(\beta E_{T} - P_{T-1})))^2}{\beta \lambda^2 + c} \]

\[ \frac{2(\beta E_{T} - P_{T-1} + bc + b\beta \lambda^2)}{\beta \lambda^2 + c} S_{T-2} - \frac{2S_{T-2}}{\beta \lambda^2 + c} \]

\[ \text{Cov}(P_{T-1}, P_{T-1}(\beta E_{T} - P_{T-1}))) + K_1 \]

where \( K_1 \) is a constant term independent of \( S_{T-2} \) and \( E_{T-1} = E(P_{T-1}) \), therefore the first order condition for (A2) is

\[ \beta (E_{T-1} - \lambda \sigma^2 S_{T-2}^*) + \lambda (bc + b\beta \lambda^2) \sigma^2 + \frac{\lambda^2}{\beta \lambda^2 + c} \text{Cov}(P_{T-1}, P_{T-1}(\beta E_{T} - P_{T-1}))) - P_{T-2} - \frac{c(S_{T-2} - b)}{\beta \lambda^2 + c} = 0 \]

\[ \Rightarrow (\beta \lambda^2 + c)S_{T-2}^* = \beta E_{T-1} - P_{T-2} + \frac{bc + b\beta \lambda^2}{\beta \lambda^2 + c} + \frac{\beta \lambda^2 + c}{\beta \lambda^2 + c} \text{Cov}(P_{T-1}, P_{T-1}(\beta E_{T} - P_{T-1}))) \]

\[ \Rightarrow S_{T-2}^* = \frac{\beta E_{T-1} - P_{T-2} + \frac{bc}{\beta \lambda^2 + c} (1 + \frac{\beta \lambda^2}{\beta \lambda^2 + c} + \frac{\beta \lambda^2 + c}{\beta \lambda^2 + c} \text{Cov}(P_{T-1}, P_{T-1}(\beta E_{T} - P_{T-1})))}{\beta \lambda^2 + c} \]

\[ \frac{b(\beta \lambda^2)^2 + \lambda \beta}{\beta \lambda^2 + c} \text{Cov}(P_{T-1}, P_{T-1}(\beta E_{T} - P_{T-1}))) \]

In general, define \( f_0(P_t) = \text{Cov}(P_t, P_t(\beta E_{t+1} - P_t)), \forall t \) and \( f_k(P_t) = \text{Cov}(P_t, P_t f_{k-1}(P_{t+1})), \forall k = 1, 2, ..., \forall t \). Then we can state the following theorem:

**Theorem 1**

The optimal speculative stock level which solves (A) under the assumption that \( T \) is the terminal date is:

\[ S_t^* = \frac{\beta E_{t+1} - P_t + bc}{\beta \lambda^2 + c} T_{t+1} (\frac{\beta \lambda^2}{\beta \lambda^2 + c})^1 + b(\beta \lambda^2) T_{t+1} (\frac{\beta \lambda^2}{\beta \lambda^2 + c})^2 \]

\[ + \frac{\beta \lambda}{\beta \lambda^2 + c} T_{t+1} (\frac{\beta \lambda}{\beta \lambda^2 + c})^2 \text{Cov}(P_{t+1}, P_{t+1}) \]

\[ \forall t = 0, 1, 2, ..., T-1. \]

**Proof**

The cases \( t = T-1 \), \( T-2 \) can be checked easily. Now, assume at time \( t \), \( S_t^* \) satisfies eq. (1). Then at time \( (t-1) \) the speculator’s problem is

\[ (A3) \max \beta [E(P_t(S_{t-1} - S_{t-1}^*)) - \frac{\lambda}{2} \text{Var}(P_t(S_{t-1} - S_{t-1}^*))) + \frac{c(S_{t-1} - b)^2}{2} + d] \]

\[ + \text{Var}(P_t(S_{t-1} - S_{t-1}^*)) \]

where \( K_2 \) is a constant term independent of \( S_{t-1}^* \). From eq. (1), we have

\[ \text{Var}(P_t(S_{t-1} - S_{t-1}^*)) = \text{E}(P_{t} - E_{t}) S_{t-1} (\frac{bc}{\beta \lambda^2 + c}) T_{t+1} (\frac{\beta \lambda^2}{\beta \lambda^2 + c})^1 (P_{t} - E_{t}) - \frac{b(\beta \lambda^2)}{\beta \lambda^2 + c} T_{t+1} (P_{t} - E_{t}) (\frac{\beta \lambda^2}{\beta \lambda^2 + c})^2 \text{Cov}(P_{t+1}, P_{t+1}(\beta E_{t+1} - P_{t+1})) \]

\[ + \frac{\beta \lambda}{\beta \lambda^2 + c} T_{t+1} (\frac{\beta \lambda}{\beta \lambda^2 + c})^2 \text{Cov}(P_{t+1}, P_{t+1}(\beta E_{t+1} - P_{t+1})) \]

\[ + \text{Var}(P_t(P_{t+1} - P_t)) \]

In general, define \( f_0(P_t) = \text{Cov}(P_t, P_t(\beta E_{t+1} - P_t)), \forall t \) and \( f_k(P_t) = \text{Cov}(P_t, P_t f_{k-1}(P_{t+1})), \forall k = 1, 2, ..., \forall t \). Then we can state the following theorem:
\[
= \sigma^2 T^{-1} - \frac{2b\lambda^2}{\beta\lambda^2 + c} \sum_{i=1}^{T-1} d_i S_{t-1} - 2b(-\frac{\beta\lambda^2}{\beta\lambda^2 + c})T^{-1} - \frac{2}{\beta\lambda^2 + c} \text{Cov}(P_t - P_{t+1}, -P_t) - \\
\frac{2}{\beta\lambda^2 + c} \sum_{i=0}^{T-2} \frac{2b\lambda^2}{(\beta\lambda^2 + c)^2} f_i^2 \text{Cov}(P_{t-i+1}, f_i(P_{t+1})) \\
+ \sigma^2 S_{t-1} - \frac{2b\lambda^2}{\beta\lambda^2 + c} \sum_{i=0}^{T-1} d_i S_{t-1} - 2b\lambda T^{-2} - \frac{2}{\beta\lambda^2 + c} f_0(P_t) - \\
\frac{2b\lambda^2}{(\beta\lambda^2 + c)^2} \sum_{i=0}^{T-2} \frac{2b\lambda^2}{(\beta\lambda^2 + c)^2} f_i^2 f_i(P_t)
\]

where \(d = \beta\lambda^2/(\beta\lambda^2 + c)\). Therefore, the first order condition for (A3) is:

\[
\beta(E_t - \lambda^2 S_{t-1}) + \frac{b\lambda^2}{\beta\lambda^2 + c} \sum_{i=0}^{T-1} d_i + \frac{\lambda^2}{\beta\lambda^2 + c} f_0(P_t) + \\
\frac{\beta^2}{(\beta\lambda^2 + c)^2} \sum_{i=0}^{T-2} \frac{2b\lambda^2}{(\beta\lambda^2 + c)^2} f_i^2 f_i(P_{t+1}) - P_{t-1} - (S_{t-1} - b) = 0
\]

\(S^*_t = \frac{\beta E_t - P_{t-1}}{\beta\lambda^2 + c} + \frac{b\lambda^2}{\beta\lambda^2 + c} + \frac{\lambda^2}{\beta\lambda^2 + c} \sum_{i=0}^{T-1} d_i + \frac{\lambda^2}{\beta\lambda^2 + c} f_0(P_t) + \\
\frac{\beta^2}{(\beta\lambda^2 + c)^2} \sum_{i=0}^{T-2} \frac{2b\lambda^2}{(\beta\lambda^2 + c)^2} f_i^2 f_i(P_{t+1})
\]

\(S^*_t = \frac{\beta E_t - P_{t-1}}{\beta\lambda^2 + c} + \frac{b\lambda^2}{\beta\lambda^2 + c} + \frac{\lambda^2}{\beta\lambda^2 + c} \sum_{i=0}^{T-1} d_i + \frac{\lambda^2}{\beta\lambda^2 + c} f_0(P_t) + \\
\frac{\beta^2}{(\beta\lambda^2 + c)^2} \sum_{i=0}^{T-2} \frac{2b\lambda^2}{(\beta\lambda^2 + c)^2} f_i^2 f_i(P_{t+1})
\)

by letting \(K = i + 1\). Therefore, the proof is completed.

G.E.D.

Eq. (1) expresses the optimal speculative stock level as the summation of four terms: (1) the current expected profit effect \(\beta E_t - P_t\); (2) the terminal convenience yield effect \(b(-\frac{\beta\lambda^2}{\beta\lambda^2 + c})T^{-1}\); (3) the cost-factor-and-convenience-yield interaction effect \(\frac{b\lambda^2}{\beta\lambda^2 + c} \sum_{i=0}^{T-1} f_i(P_{t+1})\); and (4) the covariance risk effect \(\frac{\beta\lambda^2}{(\beta\lambda^2 + c)^2} \sum_{i=0}^{T-1} f_i(P_{t+1})\).

Among these four effects, (2) vanishes as \(T\) approaches infinity, while (3) and (4) are special features arising because the mean-variance approach is used to describe the speculator's preferences. These will be discussed further in the following section.

IV. Properties of the Optimal Stock Level

First, note that in the derivations leading to eq. (1), we implicitly assumed \(S^*_t \geq 0\). However, since short selling is not allowed, the optimal speculative stock level should be written as \(S^*_t = \max(S^*_t, 0)\) for every \(t \geq 0\); and if we have \(t'\), such that \(S^*_t \leq t'\), then all formulas for \(S^*_t \leq t'\) now are invalid. This introduces a complex discontinuity into the problem. In the general case, we will simply assume \(S^*_t \leq 0\). [There are some special cases, however, in which \(S^*_t \leq 0\) can be proved (i.e., the case where \(c = 0\)].

Second, assume \(\lambda = 0\) and let \(T\) approach infinity. Then eq. (1) reduces to the case considered in [4], i.e., all competitive speculators are expected-profit maximizing agents. Hence, eq. (1) becomes \(S^*_t = \frac{\beta E_{k+1} - P_k}{c} + b\) since the terminal convenience yield effect (2) and covariance risk effect (4) both vanish when \(\lambda = 0\),
T → ∞, and the interaction effect becomes b. There is one period
time-lag difference between our model and that used in [4] as to when
the inventory cost occurs. Adjusting for this, we obtain the optimal
stock level derived in [4], which is therefore a special case of our
model.

Third, note that eq. (1) holds when T is the terminal date.
But to solve (A), we must let T approach infinity which creates
convergence problems. Note that if c > 0, then 0 < d = \frac{\beta \lambda \sigma^2}{\beta \lambda \sigma^2 + c} < 1,
and convergence problems only arise from the covariance risk effect.
However, if c = 0 (i.e., there are no variable inventory costs), then
all the terms except (2) require further consideration.

The last points we want to make are about the interaction
effect and the covariance risk effect. Each of these is a discounted
sum of a sequence but using apparently different discount rates.
\frac{\beta \lambda \sigma^2}{\beta \lambda \sigma^2 + c} and \frac{\beta \lambda}{\beta \lambda \sigma^2 + c}

When we introduce \( f_k(\cdot) \) into eq. (1), it turns out that both
expressions involve the same discount rate \( d = \frac{\beta \lambda \sigma^2}{\beta \lambda \sigma^2 + c} \). If we let \( b = 0 \) or \( c = 0 \), then the interaction effect vanishes (but the
 covariance risk effect remains). As for \( f_k(\cdot) \), these functions all
 take the covariance operator form. For example,

\[ f_0(P_t) = \text{Cov}(P_t, P_t \Delta E_{t+1} - P_t) \]

\[ f_k(P_t) = \text{Cov}(P_t, P_t f_{K-1}(P_{t+1})) \]

measures the covariance between price and expected profit \( K \) periods
later (by updating information at each subsequent future period).
Therefore, we named (4) as the covariance risk effect. Note that this
effect comes across time, rather than across alternatives at a point
in time (which leads to a covariance risk effect in the Capital Asset
Pricing Models).

V. Linear Expectation Rule

Now, assume every speculator is identical with price
expectation formation equation given by:

\[ P_t^e = \delta + \alpha P_{t-1} + \epsilon_t, \quad \forall t \]

(2)

where \( \alpha \) is the price expectation adjustment coefficient, \( \delta/(1 - \alpha) \) is
the long-run rational expectations equilibrium price. \( \{ \epsilon_t \} \) is a
sequence of identically independently distributed random variables
with \( E(\epsilon_t | P_{t-1}) = 0 \), \( \text{Var}(\epsilon_t | P_{t-1}) = \sigma^2 \) and \( E(\epsilon_t^3 | P_{t-1}) = 0 \) (i.e.,
the probability density function of \( \epsilon_t \) is symmetric with respect to zero).
\( \forall t \). When \( \alpha > 1 \), we say the speculator is responsive; when \( \alpha < 1 \), we
say he is unresponsive.

Using (2), we can determine \( f_k(P_t) \) for every \( K \geq 0 \). For
example,

\[ f_0(P_t) = \text{Cov}(P_t, P_t \Delta E_{t+1} - P_t) = \text{Cov}(P_t, P_t (\beta \delta + (\beta \alpha - 1) P_t)) \]
\[ \begin{align*}
= \beta \delta^2 + (\beta a - 1) \text{Cov}(P_t \cdot P_t) \\
= \beta \delta^2 + (\beta a - 1) \mathbb{E}[\{2(\delta + aP_{t-1})e_t + e_t^2 - \sigma^2\}]
\end{align*} \]

also,

\[ f^*_1(P_t) = \text{Cov}(P_t \cdot P_t f^*_0(P_{t+1})) = \text{Cov}(P_t \cdot P_t (\beta \delta + 2(\beta a - 1)(\delta + aP_t))\sigma^2) \]

\[ = \sigma^2 [\beta \delta + 2(\beta a - 1)\delta + 2\sigma^2(\beta a - 1)\text{Cov}(P_t \cdot P_t)] \]

\[ = (\beta \delta + 2(\beta a - 1)\delta + 2a(\beta a - 1)(\delta + aP_{t-1}))\sigma^4 \]

In general, we can prove the following theorem:

**Theorem 2**

Under the linear expectation rule (eq. (2)),

\[ f^*_k(P_t) = (\beta \delta + 2\delta(\beta a - 1)) \sum_{j=1}^{K+1} a^{j-1} + \]

\[ 2a^{K+1}(\beta a - 1)P_{t-1}\sigma^2(\delta + aP_{t-1}) \quad \forall K, \forall P_t. \quad (3) \]

[Proof]

The cases where \( K = 0 \) and \( K = 1 \) can be easily checked. Now, assume for \( K \neq 1 \), \( f^*_k(P_t) \) satisfies eq. (3), hence

\[ f^*_{k+1}(P_t) = \text{Cov}(P_t \cdot P_t f^*_k(P_{t+1})) \]

\[ = \text{Cov}(P_t \cdot P_t (\beta \delta + \delta) \sum_{j=1}^{K+1} 2(\beta a - 1)\alpha^{j-1} + 2\alpha^{K+1}(\beta a - 1)P_t)\sigma^{2+2}) \]

\[ = (\beta \delta + \delta) \sum_{j=1}^{K+1} 2(\beta a - 1)\alpha^{j-1}\sigma^{2+4} + 2\alpha^{K+1}(\beta a - 1)\sigma^{2+2}\text{Cov}(P_t \cdot P_t^2) \]

\[ = (\beta \delta + \delta) \sum_{j=1}^{K+1} 2(\beta a - 1)\alpha^{j-1}\sigma^{2+4} + 2(\beta a - 1)\alpha^{K+1}\sigma^{2+4}(\delta + aP_{t-1}) \]

\[ = \{\beta \delta + 2\delta(\beta a - 1)\} \sum_{j=1}^{K+1} a^{j-1} + 2(\beta a - 1)\alpha^{K+1}\sigma^{2+4}(\delta + aP_{t-1}) \]

which completes the proof.

Q.E.D.

Substituting eq. (3) into eq. (1), we have

\[ S^*_t = \frac{\beta \delta t + 1 - P_t}{\beta \lambda \sigma^2 + c} + \frac{bc}{\beta \lambda \sigma^2 + c} \sum_{i=0}^{T-2-1} \left( \frac{\beta \lambda \sigma^2}{\beta \lambda \sigma^2 + c} \right)^i + b\left( \frac{\beta \lambda \sigma^2}{\beta \lambda \sigma^2 + c} \right)^{T-2} + \]

\[ \frac{\beta \lambda \sigma^2}{\beta \lambda \sigma^2 + c} \sum_{i=0}^{T-2-1} \left( \frac{\beta \lambda \sigma^2}{\beta \lambda \sigma^2 + c} \right)^i. \]

\[ \left( \beta \delta + 2\delta(\beta a - 1)\sum_{j=1}^{K+1} a^{j-1} + 2a^{K+1}(\beta a - 1)P_t \right). \quad (4) \]

The next problem we consider is conditions under which \( S^*_t \) will converge as \( T \to \infty \).

VI. **Zero Variable Inventory Cost**

When \( c = 0 \), eq. (4) becomes

\[ S^*_t = \frac{\beta \delta t + 1 - P_t}{\beta \lambda \sigma^2} + b \sum_{i=0}^{T-2-1} \left( \frac{\beta \delta + 2\delta(\beta a - 1)}{\beta \lambda \sigma^2} \right)^i + \frac{2a^{K+1}(\beta a - 1)P_t}{\beta \lambda \sigma^2} \]
\[
\begin{align*}
\mathbb{E}_{t+1} - P_t &= \frac{\beta \mathbb{E}_{t+1} - P_t}{\beta \sigma^2} + b + \frac{T-t-2}{\beta \sigma^2} \left( \sum_{i=0}^{\beta \sigma} (\beta \sigma - 1) \cdot (1 - a) \cdot \frac{2(\beta \sigma - 1)(i+1)}{\beta \sigma^2} + \frac{2(\beta - 1)P_t}{\beta \sigma^2} \right), \\
&= \mathbb{E}_{t+1} - P_t + b + \frac{T-t-2}{\beta \sigma^2} \left( \sum_{i=0}^{\beta \sigma} (\beta \sigma - 1) \cdot (1 - a) \cdot \frac{2(\beta \sigma - 1)(i+1)}{\beta \sigma^2} + \frac{2(\beta - 1)P_t}{\beta \sigma^2} \right)
\end{align*}
\]

if \( a \neq 1 \)

(4')

**Theorem 3**

Given \( c = 0 \), assume \( \delta \neq 0 \). Then if \( T \to \infty \), \( S_t^* \) is unbounded for every \( t \).

[Proof]

Obviously, when \( a = 1, \delta \neq 0, T \to \infty \), then \( S_t^* \to \infty \). On the other hand, if \( a \neq 1 \), then for \( S_t^* \) to be bounded, we must require that:

(1) \( a < 1 \) and (2) \( \lim_{t \to \infty} \frac{\delta \mathbb{E}_{t+1} - P_t}{\beta \sigma^2} + 2(\beta \sigma - 1)(1 - a) + \frac{2(\beta - 1)P_t}{\beta \sigma^2} = 0 \). Now, (2) implies \( \beta(1 - a) + 2(\beta \sigma - 1) = 0 \Rightarrow \beta(a + 1) = 2 \), contradicting \( a, \beta < 1 \). Hence \( S_t^* \) is unbounded when \( \delta \neq 0, T \to \infty \).

Q.E.D.

Although \( S_t^* \) is unbounded from below, yet since short selling is prohibited, \( S_t \) must be non-negative. Hence the optimal stock \( \hat{S}_t = \max(S_t^*, 0) \) is either \( = 0 \) or \( 0 \) when \( \delta \neq 0 \) and \( T \to \infty \). This implies \( \hat{S}_t = 0 \), \( \forall t \).

**Corollary 1**

Given \( c = 0, \delta \leq \hat{S}_t \leq 0, \forall t \) and \( \hat{S}_t > 0 \) for some \( t \) implies one of the following conditions:

(i) \( \delta = 0, a < 1 \)

(ii) \( T \) is finite.

**Corollary 2**

Given \( c = 0, \delta = 0, a < 1, \) and \( T \to \infty \) implies \( \hat{S}_t = 0, \forall t \).

Corollary 1 and 2 show that with zero variable inventory holding costs, when \( T \to \infty \), and short selling is prohibited, then the speculator either accumulates unbounded stocks or no stocks at all, i.e., speculators are either highly active or totally inactive. For example, when \( a = 0, a < 1 \), they always hold zero stock. In the other cases, when \( \delta \neq 0 \), they might switch from an unbounded stock to zero at some points, and then remain for few periods, finally they switch from zero to an unbounded level of stocks. This implies they are highly active.

**Theorem 4**

Given \( c = 0 \), when \( T \to \infty \), any time-independent linear expectation of speculators won't be fulfilled.

The proof of Theorem 4 involves the structure of non-speculative excess demand, therefore we'll put it into the Appendix after the introductions of market demand structure. Nonetheless, the reason we state Theorem 4 here is to claim that \( T \to \infty \) is also not a
useful assumption to avoid the "unboundedness" problems that arise when \( c = 0 \). In the following sections, \( c \neq 0 \) is assumed.

VII. Properties of the Optimal Stock Level

When \( c \neq 0 \), eq. (4) can be written as:

\[
S_t^* = \frac{\mathbb{E}_t(T) - P_t}{\beta \lambda c^2 + c} + M_{1t} + M_{2t} + M_{3t} + M_{4t} + M_{5t}
\]

where

\[
M_{1t} = \frac{bc}{\beta \lambda c^2 + c} \cdot \frac{1 - \varphi - T_t - t}{1 - \varphi},
\]

\[
M_{2t} = \frac{b \varphi - T_t}{1 - \varphi},
\]

\[
M_{3t} = \frac{\beta^2 \lambda c^2}{\beta \lambda c^2 + c} \cdot \frac{1 - \varphi - T_t - T_t - t}{1 - \varphi},
\]

\[
M_{4t} = \begin{cases}
\frac{25(\beta a - 1) \beta \lambda c^2}{(\beta \lambda c^2 + c)^2} \cdot \varphi - T_t - T_t - t, & \text{when } a \neq 1 \\
25(\beta a - 1) \beta \lambda c^2 \varphi - T_t - T_t - t, & \text{when } a = 1
\end{cases}
\]

\[
M_{5t} = \frac{2 \beta a \beta c - 1) \beta \lambda c^2}{(\beta \lambda c^2 + c)^2} \cdot \varphi - T_t - T_t - t - 1
\]

and

\[
\varphi = \frac{\beta^2 \lambda c^2}{\beta \lambda c^2 + c}
\]

Hence, as \( T \to \infty \), \( M_{1t} \to \frac{bc}{\beta \lambda c^2 + c} \cdot \frac{1}{1 - \varphi} = b; M_{2t} \to 0; M_{3t} \to \frac{\beta^2 \lambda c^2}{\beta \lambda c^2 + c} \cdot \frac{1}{1 - \varphi} = \frac{\beta^2 \lambda c^2}{c(\beta \lambda c^2 + c)}
\]

and

\[
M_{4t} \to \begin{cases}
\frac{25(\beta a - 1) \beta \lambda c^2}{c(1 - \alpha)(\beta \lambda c^2 + c)} \cdot \frac{1 - \alpha}{(1 - \alpha)(\beta \lambda c^2 + c)[(1 - \alpha)(\beta \lambda c^2 + c)]}, & \text{when } a \neq 1 \text{ and } \frac{\beta \lambda c^2}{\beta \lambda c^2 + c} < 1 \\
25(\beta a - 1) \beta \lambda c^2 \cdot \frac{1 - \alpha}{\beta \lambda c^2 + c}, & \text{when } a = 1 \text{ and } \frac{\beta \lambda c^2}{\beta \lambda c^2 + c} < 1
\end{cases}
\]

\[
M_{5t} \to \begin{cases}
\frac{2 \beta a \beta c - 1) \beta \lambda c^2}{c(\beta \lambda c^2 + c)} \cdot \varphi - T_t - T_t - t, & \text{when } a \neq 1 \text{ and } \frac{\beta \lambda c^2}{\beta \lambda c^2 + c} < 1 \\
25(\beta a - 1) \beta \lambda c^2 \cdot \varphi - T_t - T_t - t, & \text{when } a = 1
\end{cases}
\]

Note that when \( a = 1 \), \( \frac{\beta \lambda c^2}{\beta \lambda c^2 + c} < 1 \) is satisfied. Therefore, we have the following theorem:

Theorem 5

Assume \( T \to \infty \). If \( \frac{\beta \lambda c^2}{\beta \lambda c^2 + c} \leq 1 \), then \( S_t^* \) is unbounded.

Furthermore, whether \( S_t^* = \infty \) or \( -\infty \) depends on \( M_{4t} \) and \( M_{5t} \). On the other hand, if \( \frac{\beta \lambda c^2}{\beta \lambda c^2 + c} < 1 \), then as \( T \to \infty \), \( S_t^* \) will converge to

\[
\overline{S_t} = \frac{(\beta a - 1) \beta \lambda c^2}{c(1 - \alpha)(\beta \lambda c^2 + c)} \cdot \varphi - T_t - T_t - t - 1 + \frac{bc}{\beta \lambda c^2 + c} \cdot \frac{1}{1 - \varphi} = b;
\]

where

\[
M = \frac{bd}{\beta \lambda c^2 + c} + \frac{\beta^2 \lambda c^2}{c(\beta \lambda c^2 + c)} + b
\]
$2\beta(a - 1)\beta\sigma^2 - \frac{2\beta(a - 1)\beta\sigma^2}{c(1 - a)(\beta\sigma^2 + c)} - \frac{2\beta(a - 1)\beta\sigma^2}{c(1 - a)(\beta\sigma^2 + c)[(1 - a)\beta\sigma^2 + c]}$

when $a \neq 1$;

and

$M = \frac{\beta}{\beta\sigma^2 + c} + \frac{\beta^2\beta\sigma^2}{\beta\sigma^2 + c} + b + \frac{2\beta(a - 1)\beta\sigma^2}{c^2}$

when $a = 1$.

Corollary 3

Given $\alpha\beta\sigma^2 < \beta\sigma^2 + c$, $\overline{S}_t > 0$ implies

(i) $\frac{\partial}{\partial h} = 1$, $\forall t$.

(ii) $\text{sgn}(\frac{\partial S_t}{\partial P_t}) = \text{sgn}(\beta a - 1)$, $\forall t$.

[Proof]

(i) is obvious. For (ii), if $a < 1$, then

$\text{sgn}(\frac{\partial S_t}{\partial P_t}) = \text{sgn}(\beta a - 1) < 0$, since $1 - a > 0$ and $\beta$, $\sigma$, $\alpha > 0$.

If $a > 1$, then since $\alpha\beta\sigma^2 < \beta\sigma^2 + c \Rightarrow c + (1 - a)\beta\sigma^2 > 0$, hence

$\text{sgn}(\frac{\partial S_t}{\partial P_t}) = \text{sgn}(\beta a - 1)$, $\forall t$.

Q.E.D.

Theorem 5 shows that $\overline{S}_t$ is the solution for problem (A) when $\alpha\beta\sigma^2 < \beta\sigma^2 + c$ and $c > 0$ (note that, by hypothesis, $\overline{S}_t > 0$, $\forall t$), therefore the optimal speculative stock level is fully characterized.

Otherwise, we always have $\overline{S}_t = 0$ or $0$. Corollary 3 shows that as the minimum-cost stock level $b$ changes by one unit, the optimal stock level $\overline{S}_t$ also changes by one unit in the same direction for every $t$.

Furthermore, when the current price $P_t$ changes, which direction $\overline{S}_t$ will change is determined by the sign of $(\beta a - 1)$.

VIII. Market Price Behavior

Now, since we take the behavior of non-speculators as given, we can summarize their impacts on the market by a non-speculative excess demand function. Following [4], we postulate a linear non-speculative excess demand function of the form:

$D_t = -\alpha P_t + \gamma_t$, $a > 0$

(5)

where $(\gamma_t)$ is a sequence of identically independently distributed random variables with $E(\gamma_t) = \mu$, $\text{Var}(\gamma_t) = \nu$.

By the market clearing condition, we have

$\overline{S}_{t-1} - \overline{S}_t = -\alpha P_t + \gamma_t$, $\forall t$

$\Rightarrow zP_{t-1} - zP_t = -\alpha P_t + \gamma_t$, $\forall t$

(6)

$\Rightarrow P_t = \frac{z}{z - a}P_{t-1} - \frac{\gamma_t}{z - a}$, $\forall t$

$\Rightarrow P_t = (\frac{z}{z - a})^tP_0 - \frac{\sum_{j=0}^{t-1} (\frac{z}{z - a})^j \gamma_{t-j}}{z - a}$, $\forall t = 0, 1, 2, \ldots$

(7)

where $z = (\beta a - 1)/(1 + \alpha\beta\sigma^2 + c)$. Note that this result is derived when $\alpha\beta\sigma^2 < \beta\sigma^2 + c$ and $0 < \overline{S}_t < \infty$, $\forall t$.

From (7), we have

$\overline{S}_t = (\frac{z}{z - a})^tP_0 - \frac{\sum_{j=0}^{t-1} (\frac{z}{z - a})^j \mu}{z - a}$

$= (\frac{z}{z - a})^tP_0 - \frac{\mu}{z - a} \cdot \frac{1 - (\frac{z}{z - a})^t}{1 - \frac{z}{z - a}}$
\[ w^t P_0 + \frac{z}{a} (1 - w^t), \text{ where } w = \frac{z}{z - a}; \]

\[
\text{Var } P_t = \text{Var}(\frac{z}{z - a} w^t P_0 - \sum_{j=0}^{t-1} \frac{z}{z - a} \frac{1}{z} w^{t-j-1}) = \frac{V}{z - a} \sum_{j=0}^{t-1} (\frac{z}{z - a})^{2j} = \frac{(1 - w)^2 V}{a} \frac{1 - w^{2t}}{1 - w^2}, \]

Since \( w = \frac{z}{z - a} \Rightarrow za - aw = z \Rightarrow z = \frac{aw}{w - 1} \Rightarrow z - a = \frac{a}{w - 1}. \)

\[
\text{Cov}(P_t, P_{t-h}) = \text{Cov}(\sum_{j=0}^{t-1} \frac{z}{z - a} \frac{1}{z} w^{t-j-1}, \sum_{k=0}^{t-h-1} \frac{z}{z - a} \frac{1}{z} w^{k-h-k}) = \sum_{k=0}^{t-h-1} \frac{V}{z - a} \frac{(1 - w)^2 V}{a} \frac{1 - w^{2t-2h}}{1 - w^2}, \text{ when } 0 \leq h < t.
\]

This proves:

**Theorem 6**

Given \( c \neq 0, \alpha \lambda \sigma^2 < \beta \lambda \sigma^2 + c, 0 \leq S_t \leq \Psi t, \)

(i) \( \lim_{t \to \infty} E P_t = \begin{cases} \frac{z}{z - a}, & \text{if } |w| < 1 \\ \infty, & \text{if } |w| > 1 \end{cases} \)

(ii) \( \lim_{t \to \infty} \text{Var } P_t = \begin{cases} \frac{(1 - w)^2 V}{a^2 (1 + w)}, & \text{if } |w| < 1 \\ \infty, & \text{if } |w| > 1 \end{cases} \)

(iii) \( \lim_{t \to \infty} \text{Cov}(P_t, P_{t-h}) = \begin{cases} \frac{w^h (1 - w)^2 V}{a^2 (1 + w)}, & \text{if } |w| < 1 \\ \infty, & \text{if } |w| > 1 \end{cases} \)

Since we want price to be non-negative, we make the following assumptions: (i) \( w > 0 \) and (ii) \( P_0 > \frac{z}{a}. \) Therefore, for \( |w| < 1, \) we need \( 0 < \frac{z}{z - a} < 1 \Rightarrow z < 0 \Rightarrow \beta a < 1, \) since \( (1 - a)\beta \lambda \sigma^2 + c > 0. \)

Now, if \( c > (1 - \beta)\lambda \sigma^2, \) then \( \beta \lambda \sigma^2 + c > \lambda \sigma^2 \Rightarrow 1 + \frac{c}{\beta \lambda \sigma^2} > \frac{1}{\beta}, \) hence \( a < \frac{\beta \lambda \sigma^2 + c}{\beta \lambda \sigma^2} \Rightarrow a < \frac{1}{\beta} \) which establishes the following:

**Theorem 7**

Assume \( a > 1. \) If \( c > (1 - \beta)\lambda \sigma^2, \) then \( S_t \) bounded and \( \lim_{t \to \infty} E P_t, \lim_{t \to \infty} \text{Var } P_t \) unbounded do not violate market clearing. Under this configuration, the action of competitive speculators will destabilize prices.

As to whether the speculator's expectations will be fulfilled, we can compare eq. (6) and eq. (2) to derive the following theorem:

**Theorem 8**

Fulfilling of speculator's expectation implies:

(i) \( \frac{z}{z - a} = a \) and

(ii) \( \frac{\gamma_t}{a - z} = \delta + \epsilon_t \) where

\[ z = \frac{\beta a - 1}{(1 + a)\beta \lambda \sigma^2 + c}, \]
\[ \beta \lambda \sigma^2 + c (1 - a)\beta \lambda \sigma^2 + c \]
and (ii) holds for every \( t = 1, 2, \ldots \).

**Corollary 4**

If speculator’s expectations are fulfilled, then

(i) \( a < 1 \), (ii) \( \mu = \delta(a - z) \), (iii) \( V^2 = (a - z)^2 \sigma^2 \).

[Proof]

Assume \( a = 1 \), then fulfilling expectation implied \( \frac{z}{z - a} = 1 \)

\[ \Rightarrow z = z - a \Rightarrow a = 0, \] contradiction. On the other hand, if \( a > 1 \),

then \( \frac{a}{z - a} = z > a > 0 \) and \( \frac{\delta S_t}{\delta P_t} = z > a \). Therefore, as

\[ P_t \rightarrow \infty, \quad S_t \rightarrow \infty \] which is unbounded. Since we only dealt with bounded

\( S_t \), hence \( a < 1 \) is required. (ii) and (iii) are derived from

\[ E\left( \frac{\gamma_t}{z - a} \right) = E(\delta + \epsilon_t) \text{ and } \text{Var}\left( \frac{\gamma_t}{z - a} \right) = \text{Var}(\delta + \epsilon_t), \] respectively (where expectations are conditional on available information).

Q.E.D.

Therefore, when speculator’s expectations are fulfilled, \( EP_t, \)

\( \text{Var} P_t \) and \( \text{Cov}(P_t, P_{t-1}) \) are all bounded. Also, \( S_t^* \) is bounded.

**IV. Profitable Speculation**

In this section, we turn to Friedman’s conjecture, i.e.,

profitable speculation necessarily stabilizes prices. Recall that, in

problem (A), \( S_t = b \), \( \forall t \) is a feasible strategy, therefore, any

strategy \( \{S_t\} \) with \( S_t \neq b \) for some \( t \) certainly incurs positive profits

(actually, the profits must be high enough to cover the losses from
change in variances). From Theorem 5, this implies \( M + z P_t \neq b \) for
some \( t \) which is easily to be satisfied.

Now, from Theorem 7, when \( a > 1 \), there would be destabilizing
profitable speculation. However, in this case, the speculator’s
expectations won’t be fulfilled. On the other hand, when their
expectations are fulfilled, \( a < 1 \) and

\[ \text{Var} P_t = \frac{(1 - w) V}{a^2(1 + w)} \cdot \left(1 - w^2 t\right) < \frac{V}{a^2} \]

(since \( a < 1 \) \( \Rightarrow z < 0 \) \( \Rightarrow 0 < w = \frac{z}{z - a} < 1 \)), where \( \frac{V}{a^2} = \text{Var} P_t \) when

there are no speculators. Therefore,

**Theorem 9**

At a rational expectations equilibrium (i.e., speculators’
expectations are fulfilled), and given a linear non-speculative excess
demand, profitable speculation always stabilizes prices.

Theorem 9 leaves it open whether at a rational expectations
equilibrium with non-linear non-speculative excess demand, profitable
speculation always stabilizes prices. Because of earlier results (see
[1], [3], [6]), it seems unlikely that Friedman’s conjecture will hold
with non-linear excess demands, however.
Y. Conclusion

In this paper, speculators are taken to be risk averse, and a mean-variance approach was employed. Under this approach, the optimal stock level for speculators has been derived. Nonetheless, this stock level might be unbounded. To carry the analysis further, we found when marginal inventory cost is zero, speculators are either highly active \( (S_t = \infty) \) or inactive \( (S_t = 0) \). To resolve the problem of unboundedness of \( S_t \) when \( c = 0 \) requires either the assumption that the long-run equilibrium price equals zero (which leads to \( S_t = 0, \forall t \)) or the assumption of finite horizon (in which case speculators’ expectations won’t be fulfilled\(^5\)).

On the other hand, when the inventory carrying cost function is of a non-degenerate quadratic form, one possible equilibrium configuration involves bounded stock levels and unbounded prices, with the expectation adjustment coefficient greater than 1. However, this won’t constitute a rational expectations equilibrium.\(^6\) When a rational expectations equilibrium exists given linear non-speculative excess demand, the stock level is bounded, price is also bounded, and Friedman’s conjecture is verified, i.e., profitable speculation necessarily stabilizes prices.

Appendix: Proof of Theorem 4

By inspecting eq. (2) and eq. (4'), we know that unless \( \beta \alpha - 1 = 0 \), speculator’s expectations won’t be fulfilled, since in (4'), price terms involve multiplicative time factors when \( \beta \alpha \neq 1 \). On the other hand, when \( \beta \alpha - 1 = 0 \), then

\[
S^*_t = \frac{\beta \lambda}{\beta \lambda \sigma^2} b + \frac{\beta \lambda}{\beta \lambda \sigma^2} (T - t - 1) = \frac{\beta \lambda}{\beta \lambda \sigma^2} (T - t) + b \geq 0, \forall t \leq T
\]

\[
\Rightarrow S^*_{t-1} - S^*_t = \frac{b}{\lambda \sigma^2}, \forall t \leq T
\]

Therefore, the market clearing condition becomes

\[
-\alpha p_t + \gamma_t = \frac{b}{\lambda \sigma^2}, \forall t \leq T
\]

\[
\Rightarrow p_t = \frac{\gamma_t}{\alpha} - \frac{b}{\alpha \lambda \sigma^2}, \forall t \leq T
\]

Now, for expectations to be fulfilled, we need \( \alpha = 0 \) which contradicts \( \beta \alpha - 1 = 0 \). Hence, the proof is completed.
Footnotes

* I am indebted to James Quirk for helpful discussions and editings, also to Richard McElvee for comments on earlier drafts. All errors, of course, remain mine.

1. On the other hand, both Sarris [5] and Turnovsky [7] employed mean-variance approach to determine one-period optimal stock level, without taking account the dynamic effects.

2. The assumption \( \text{Var}(P_t | P_{t-1}) = \sigma^2 \), \( \forall t \) can be relaxed to \( \text{Var}(P_t | P_{t-1}) = \sigma_t^2 \) which is a constant term independent of \( P_t \), \( P_{t-1} \), but might change over time. Under this assumption, the results can be easily adjusted to characterize the optimal stock level. Nonetheless, the market price process will be highly complexized and difficult to proceed.

3. Note that, suppose instead of quadratic form, we use linear form for inventory cost function. Thus, when \( b = 0 \) (i.e., no convenience yield), the inventory cost curve is a straight line over \([0, \infty)\). However, if \( b \neq 0 \), then we have to introduce a kinked point in inventory cost curve.

4. Strictly speaking, the left-hand side of market clearing equation should be multiplied by the number of representative speculators, but without loss of generality, we set this factor to be one.

5. In the one-period framework, considering all the agents in the market, Turnovsky [7] showed that constant marginal inventory cost may lead to nonexistence of rational expectations equilibrium in futures market as well.

6. Without considering the "unboundedness" problem of optimal stock level, when \( a > 1 \) and the speculator's expectation is fulfilled, the optimal stock level will increase monotonically over time. Therefore, there will never be realized profits.
REFERENCES


